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Addendum to Special Report No. 170-1
on

The Relationship between Coaxial and Cylindrical Wave Guide and Cavity Modes

by
Morris Kline
1 April 1947

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Addendum to Special Report No. 170-1

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Abstract

Some details of the proofs contained in special report No. 170-1, which were based on physical considerations, are herein established by purely mathematical means.

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INTRODUCTION. In special report No. 170-1 it was proved that each transverse electric or transverse magnetic mode associated with a circularly cylindrical coaxial wave guide or cavity reduces to a corresponding mode associated with a circularly cylindrical wave guide or cavity, as the radius of the inner coaxial conductor approaches zero. This proof was based on a mathematical theorem which involves the behavior of the explicit solutions of the equations,

$$J_n(x)\ N_n(\rho x) - J_n(\rho x)\ N_n(x) = 0, \qquad (1)$$

$$J_n'(x)\ N_n'(\rho x) - J_n'(\rho x)\ N_n'(x) = 0. \qquad (2)$$

Specifically, the theorem of report No. 170-1 shows that to each root, x_m, of $J_n(x) = 0$ $\left[\text{or } x_m',\text{ of } J_n'(x) = 0\right]$ there corresponds an explicit solution, $x(\rho)$, of equation (1) $\left[\text{or (2)}\right]$ above, which approaches x_m $\left[\text{or } x_m'\right]$ as ρ approaches zero*; and conversely, that each solution of equation (1) $\left[\text{or (2)}\right]$ approaches a root of $J_n(x) = 0$ $\left[\text{or } J_n'(x) = 0\right]$. Moreover, the correspondence is one-to-one so that one can assert that the m^{th} positive root of $J_n(x) = 0$ $\left[\text{or } J_n'(x) = 0\right]$ corresponds to the m^{th} positive solution, in order of magnitude, of equation (1) $\left[\text{or (2)}\right]$, for ρ near 0. In the proof of the converse statement of the theorem, physical evidence was introduced to show that an explicit solution of equation (1) $\left[\text{or (2)}\right]$ must be continuous and cannot approach zero, nor tend to infinity, but must approach a finite limit.

The purpose of this addendum is to supply mathematical proof of those points which were previously established by physical reasoning. Before doing this it must be made clear just what is meant by a solution, $x(\rho)$, of (1) $\left[\text{or (2)}\right]$, how much we can expect to prove about such a solution, and how much we need to prove.

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*Throughout the discussion, both in the main report and in this addendum, we are interested only in values of ρ that satisfy the inequality $\rho \geq 0$.

By the implicit function theorem, under conditions which are stated in report No. 170-1 and in any advanced calculus text*, and which are satisfied by (1) [and (2)], in a sufficiently restricted neighborhood of some pair of values, $\rho = \rho_0$, $x = x_0$, the equation determines x uniquely as a continuous function of ρ. By direct application of this theorem we proved the first part of the theorem stated above, namely, that to each root, x_m, of $J_n(x) = 0$ [or x'_m, of $J'_n(x) = 0$] there corresponds a solution, $x(\rho)$ of, equation (1) [or (2)], continuous in a suitably restricted neighborhood of $\rho = 0$, $x = x_m$ [or x'_m].

With regard to the converse it is pertinent to ask what is meant by each solution of (1) [or (2)] approaching a root of $J_n(x) = 0$ [or $J'_n(x) = 0$]. We have shown that there are infinitely many continuous branches of the function $x(\rho)$, for ρ near 0; and hence it is possible to construct discontinuous solutions, $x(\rho)$, at will, by any arbitrary selection of x from these continuous branches in different parts of a range of ρ-values. Moreover, for all we know, there may be other discontinuous solutions, having no points belonging to the continuous branches. It is desired to show that, however we may construct a function, $x(\rho)$, satisfying (1) [or (2)] identically in a neighborhood of $\rho = 0$, if such a function has a finite limiting value as ρ tends to 0 that limiting value must be one of the values, x_m [or x'_m], satisfying $J_n(x) = 0$ [or $J'_n(x)$] = 0. It follows from the uniqueness of the solution, $x(\rho)$, in a sufficiently restricted neighborhood of $\rho = 0$, $x = x_m$ [or x'_m], that, in that neighborhood, the function we have constructed must belong to the particular continuous solution that tends to x_m [or x'_m]. We desire to show

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*e.g. Courant, R.: Differential and Integral Calculus, vol.II,p.114.

also, that any solution $x(\rho)$ which tends to ∞ as ρ tends to 0 must be discontinuous in every neighborhood, $0 < \rho < \varepsilon$, for any positive ε, i.e. it must tend to ∞ by an infinite sequence of jumps. That such a solution is possible is immediately evident since we can piece together a solution from the infinitely many continuous branches by choosing values from branches converging to various values of x_m $\{$or $x'_m\}$ with m tending to ∞ as ρ tends to 0.

The proof will be broken into three parts as follows:

I. $x(\rho)$ cannot have 0 as a limiting value as ρ approaches 0.

II. Any finite value approached by values of $x(\rho)$ as ρ tends to 0 must be a value of x_m $\{$or $x'_m\}$, and any value of $x(\rho)$ that exists in a sufficiently restricted neighborhood of $\rho = 0$, $x = x_m$ $[$or $x = x'_m]$ must belong to the continuous branch of $x(\rho)$ already shown to exist in that neighborhood.

III. Values of $x(\rho)$ can tend to ∞ as ρ tends to 0 only discontinuously.

We shall first prove the statements above for equation (1) and then show how the proof can be adapted to equation (2).

PROOF - Part I

Suppose $x(\rho)$ has 0 as a limiting value as ρ approaches 0. We shall show that this assumption leads to a contradiction. We first consider the case in which $n > 0$, and write equation (1) in the form

$$\frac{J_n(x)}{J_n(\rho x)} - \frac{N_n(x)}{N_n(\rho x)} = 0 \, . \qquad (3)$$

We can express $J_n(x)$ and $N_n(x)$ in the following forms:*

$$J_n(x) = \frac{1}{n!}\left(\frac{x}{2}\right)^n + R_1(x) ,$$

$$N_n(x) = -\sum_{r=0}^{n-1} \frac{(n-r-1)!}{r!}\left(\frac{x}{2}\right)^{-n+2r} + C(x)\log\frac{x}{2} + D(x) ,$$

where $R_1(x)$ is a power series in x beginning with a term in x^{n+2}, and $C(x)$ and $D(x)$ are power series in x each beginning with a term in x^n.

Equation (3) can then be written in the form:

$$\frac{\frac{1}{n!}\left(\frac{x}{2}\right)^n + R_1(x)}{\frac{1}{n!}\left(\frac{\rho x}{2}\right)^n + R_1(\rho x)} - \frac{C(x)\log\frac{x}{2} + D(x) - \sum\limits_{r=0}^{n-1}\frac{(n-r-1)!}{r!}\left(\frac{x}{2}\right)^{-n+2r}}{C(\rho x)\log\frac{\rho x}{2} + D(\rho x) - \sum\limits_{r=0}^{n-1}\frac{(n-r-1)!}{r!}\left(\frac{\rho x}{2}\right)^{-n+2r}} = 0 . \qquad (4)$$

In the first fraction we divide numerator and denominator by x^n, and in the second fraction we multiply numerator and denominator by $(\rho x)^n$. There results

$$\frac{\frac{1}{n!}\left(\frac{1}{2}\right)^n + R_2(x)}{\frac{1}{n!}\left(\frac{\rho}{2}\right)^n + R_3(\rho x)} - \frac{(\rho x)^n C(x)\log\frac{x}{2} + (\rho x)^n D(x) - \rho^n\sum\limits_{r=0}^{n-1}\frac{(n-r+1)!}{r!}\left(\frac{1}{2}\right)^{-n+2r} x^{2r}}{(\rho x)^n C(x)\log\frac{\rho x}{2} + (\rho x)^n D(\rho x) - \sum\limits_{r=0}^{n-1}\frac{(n-r+1)!}{r!}\left(\frac{1}{2}\right)^{-n+2r}(\rho x)^{2r}} = 0, \qquad (5)$$

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*See the expressions for $J_n(x)$ and $N_n(x)$ in *Introduction to Bessel Functions* by G.N.Watson,pp.40,62. The expressions for N_n and N'_n were stated in special report No, 170-1.

where $R_2(x)$ and $R_3(\rho x)$ result from dividing $R_1(x)$ and $R_1(\rho x)$ by x^n, and therefore have x^2 as the lowest power of x. If we now consider the value of the left-hand side of equation (5) when $x(\rho)$ is substituted for x and ρ approaches 0, we find that the first fraction becomes infinite while the second fraction approaches 0, for x-values near 0. The whole expression, however, should be identically 0 for all $x(\rho)$ being considered. Hence, from the contradiction we conclude that x cannot have 0 as a limiting value as ρ tends to 0.

Now considering the case in which n = 0, we again write equation (1) in the form

$$\frac{J_0(x)}{J_0(\rho x)} - \frac{N_0(x)}{N_0(\rho x)} = 0. \tag{3a}$$

Inserting the values of $N_0(x)$ and $N_0(\rho x)$, (3a) can be written

$$\frac{J_0(x)}{J_0(\rho x)} - \frac{(\gamma - \log 2 + \log x)\ J_0(x) - D(x)}{(\gamma - \log 2 + \log \rho x)\ J_0(\rho x) - D(\rho x)} = 0\ ,. \tag{6}$$

where γ is a constant and D(x) is a power series beginning with a term in x^2.

Again we assume that $x(\rho)$ has 0 as a limiting value as ρ tends to 0, and we shall show that this assumption leads to a contradiction. We first rewrite equation (6) in the form

$$\frac{J_0(x)}{J_0(\rho x)} \left[1 - \frac{\gamma - \log 2 + \log x - \frac{D(x)}{J_0 x}}{\gamma - \log 2 + \log \rho x - \frac{D(\rho x)}{J_0(\rho x)}} \right] = 0. \tag{7}$$

For x and ρx near 0, neither $J_0(x)$ nor $J_0(\rho x)$ can be 0, since $J_0(x)$ approaches 1 as x approaches 0. Hence, for the values of $x(\rho)$ that have been assumed,

$$1 - \frac{\gamma - \log 2 + \log x - \frac{D(x)}{J_0 x}}{\gamma - \log 2 + \log \rho x - \frac{D(\rho x)}{J_0(\rho x)}} = 0. \tag{8}$$

Simplifying (8)

$$\frac{\log \rho - \frac{D(\rho x)}{J_0(\rho x)} + \frac{D(x)}{J_0(x)}}{\gamma - \log 2 + \log \rho x - \frac{D(\rho x)}{J_0(\rho x)}} = 0 . \qquad (9)$$

For the assumed solution $x(\rho)$, tending to 0 with ρ, equation (9) holds identically. Hence, either the numerator is zero or the denominator is infinite.

First let us consider the numerator. For values of ρ, x, and ρx near 0, the first term will be large while the second and third terms will be near 0. Hence the numerator cannot be zero.

The denominator is dominated by the term, $\log \rho x$, and this term is infinite only when $x = 0$. Since when $\rho \neq 0$ and $x = 0$ equation (1) becomes $1 \cdot \infty - 1 \cdot \infty = 0$, these values cannot be said to satisfy (1), and hence such values need not be considered. We can therefore exclude the possibility of the denominator being infinite.

Since the assumption that $x(\rho)$ tends to 0 with ρ leads to a contradition we conclude that $x(\rho)$ cannot have 0 as a limiting value.

Part II

We now suppose that $x(\rho)$ remains bounded and has some finite limiting value as ρ tends to 0. In this case ρx also tends to 0. We consider equation (1) in the form

$$J_n(x) - \frac{N_n(x)}{N_n(\rho x)} J_n(\rho x) = 0 . \qquad (10)$$

We first note that, as a consequence of Part I, $x(\rho)$ cannot have 0 as a limiting value as ρ approaches 0. For any other finite limiting value the second term in equation (10) approaches 0 with ρ. Hence the corresponding values of $J_n(x)$ must approach 0, and since J_n is a continuous function of x, the limiting value approached by x must be a root of $J_n(x) = 0$.

Since $x(\rho)$ may be a discontinuous function of ρ there can be many limiting values approached by $x(\rho)$ as ρ tends to 0. But each must be a root of $J_n(x) = 0$. Consider the values of $x(\rho)$ which approach one particular root, x_m, of $J_n(x) = 0$. For ρ sufficiently near 0 these values all lie in an arbitrarily small interval of x-values about x_m. Moreover, we may define the value of $x(0)$ as x_m. Now the implicit function theorem asserts that any solution $x(\rho)$ of equation (1), which assumes a value $x(0)$ at $\rho = 0$ (which is x_m in this case) such that equation (1) is satisfied by $\rho = 0$, $x = x(0) = x_m$, is unique in a sufficiently small neighborhood of $(0, x_m)$. In the main report we established the existence of a continuous solution $x(\rho)$ of equation (1) which approaches any given root of $J_n(x) = 0$. It therefore follows that the values of $x(\rho)$ which we have assumed to approach x_m must belong to the continuous solution already known to exist.

We can therefore conclude that values of $x(\rho)$ can approach finite limiting values as ρ tends to 0 only along the continuous solutions found earlier. It is hence reasonable to order the continuous solutions of (1) according to the roots, x_m, of $J_n(x) = 0$, which they approach; and it follows that these continuous solutions of (1) are in one-to-one correspondence with the roots x_m.

Part III

We have finally to consider the case of an unbounded solution, $x(\rho)$, of equation (1); that is a solution which tends to infinity as ρ takes on a sequence of values approaching 0. We shall prove that such a solution can tend to infinity only by a series of discontinuous jumps.

As x becomes infinite, ρx may or may not become infinite. We consider each possibility separately, and first suppose that ρx tends to infinity as ρ tends to 0. For large values of x we may use asymptotic expansions for $J_n(x)$ and $N_n(x)$,* and write

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* The statements about the asymptotic expansions for these functions that are here used are established in Introduction to Bessel Functions by G.N.Watson, p.199.

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) + \text{terms involving higher powers}$$

of x in the denominators.

$$N_n(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) + \text{terms involving higher powers}$$

of x in the denominators.

We may write instead

$$J_n(x) - \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) = \mathcal{E}_1(x) \; ,$$

$$N_n(x) - \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) = \mathcal{E}_2(x) \; , \qquad (11)$$

where $\mathcal{E}_1(x)$ and $\mathcal{E}_2(x)$ are defined by the left-hand sides of equations (11), and both are of the order of $\frac{1}{x^{3/2}}$.

Using equations (11) we have as the exact equivalent of equation (1),

$$\left[\sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) - \mathcal{E}_1(x)\right]\left[\sqrt{\frac{2}{\pi \rho x}} \sin\left(\rho x - \frac{\pi}{4} - \frac{n\pi}{2}\right) - \mathcal{E}_2(\rho x)\right]$$

$$\left[\sqrt{\frac{2}{\pi \rho x}} \cos\left(\rho x - \frac{\pi}{4} - \frac{n\pi}{2}\right) - \mathcal{E}_1(\rho x\right]\left[\sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) - \mathcal{E}_2(x)\right] = 0.$$

Neglecting the factor $\sqrt{\frac{2}{\pi}}$, which will have no bearing on the outcome, and multiplying out,

$$\frac{1}{x\sqrt{\rho}} \cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) \sin\left(\rho x - \frac{\pi}{4} - \frac{n\pi}{2}\right) - \frac{\mathcal{E}_1(x)}{\sqrt{\rho x}} \sin\left(\rho x - \frac{\pi}{4} - \frac{n\pi}{2}\right)$$

$$- \frac{\mathcal{E}_2(\rho x)}{\sqrt{x}} \cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) + \mathcal{E}_1(x)\, \mathcal{E}_2(\rho x)$$

$$- \frac{1}{x\sqrt{\rho}} \cos\left(\rho x - \frac{\pi}{4} - \frac{n\pi}{2}\right) \sin\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) + \frac{\mathcal{E}_1(\rho x)}{\sqrt{x}} \sin\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right)$$

$$+ \frac{\mathcal{E}_2(x)}{\sqrt{\rho x}} \cos\left(\rho x - \frac{\pi}{4} - \frac{n\pi}{2}\right) - \mathcal{E}_1(\rho x)\, \mathcal{E}_2(x) = 0.$$

This may be written

$$\frac{1}{x\cdot\sqrt{\rho}}\sin(\rho x - x) + \varepsilon(x, \rho x) = 0, \tag{12}$$

where the lowest order term in $\varepsilon(x, \rho x)$ is that in $\dfrac{1}{x^2\rho\sqrt{\rho}}$.

The explicit solutions of equation (1) may be obtained from equation (12) which is its exact equivalent. Multiplying through by $x\sqrt{\rho}$, we have

$$\sin(\rho x - x) + \varepsilon'(x, \rho x) = 0,$$

where the order of $\varepsilon'(x, \rho x)$ is that of $\dfrac{1}{\rho x}$.

Hence $\rho x - x = m\pi + \arcsin \varepsilon'(x, \rho x)$,

and $$x = \frac{m\pi}{\rho - 1} + \frac{\arcsin \varepsilon'(x, \rho x)}{\rho - 1}. \tag{13}$$

Now as ρ approaches 0, we have assumed that x and ρx become infinite. Consequently ε' must tend to 0. But the continuous solutions of equation (12), and therefore of equation (1), must approach $m\pi$. Hence x can tend to infinity only by discontinuous steps. A discontinuous function, $x(\rho)$, satisfying equation (1) and tending to infinity as ρ approaches 0, can be constructed by increasing the value of m through integral steps as ρ tends to 0.

We now consider the case in which x becomes infinite as ρ takes on a sequence of values approaching 0 while ρx remains bounded. The values of ρx must then have a finite limiting value, which we shall call L.

Let us write equation (1) in the form

$$\frac{J_n(x)}{N_n(x)} - \frac{J_n(\rho x)}{N_n(\rho x)} = 0. \tag{14}$$

The fraction $\dfrac{J_n(\rho x)}{N_n(\rho x)}$ is a continuous function of ρx for $\rho x \neq 0$,

and by defining $\dfrac{J_n(0)}{N_n(0)}$ to be 0, this fraction is made continuous

for $\rho x = 0$ as well. Now let ρ take on a sequence of values tending to 0, for which x becomes infinite and ρx approaches L. Then for any positive $\mathcal{E}$, ρ can be made sufficiently small so that $\frac{J_n(\rho x)}{N_n(\rho x)}$ differs from $\frac{J_n(L)}{N_n(L)}$ by less than $\mathcal{E}$.

The fraction $\frac{J_n(x)}{N_n(x)}$, however, when $J_n(x)$ and $N_n(x)$ are replaced by expressions obtained from equations (11), becomes

$$\frac{\cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) + \mathcal{E}_1'(x)}{\sin\left(x - \frac{\pi}{2} - \frac{n\pi}{2}\right) + \mathcal{E}_2'(x)},$$

where $\mathcal{E}_1'(x)$ and $\mathcal{E}_2'(x)$ tend to 0 as x tends to infinity. This fraction approaches $\cot\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right)$ as x tends to infinity. Now if $x(\rho)$ is a continuous function of ρ for small positive values of ρ, and tends continuously to infinity as ρ tends to 0, the values of the above fraction must assume all values from $-\infty$ to $+\infty$. Hence, in view of the behavior of $\frac{J_n(\rho x)}{N_n(\rho x)}$, equation (14), and consequently equation (1), cannot be satisfied identically. In this case, as in the one considered earlier, the possibility remains that $x(\rho)$ may become infinite discontinuously so that for such values of x equation (1) may still be satisfied.

Extension of the Proof to Equation (2).

Thus far we have considered only equation (1). The proofs in the case of equation (2) are precisely the same. First, we note that J_0' and N_0' are equal to J_1 and N_1, respectively. Secondly, J_n' and N_n' have the same general properties as J_n and N_n, and as in the proof in the main report, it is merely necessary to change an exponent from n to n + 1 to carry through a proof for equation (2) just as it is carried through for equation (1).

One further change is necessary in dealing with equation (2). In Part III above asymptotic expansions were used for J_n and N_n. To obtain asymptotic expansions for J_n' and N_n' one can use the recurrence relation

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right], \tag{15}$$

together with the theorem that permits the addition of the asymptotic expansions of two functions in order to obtain an asymptotic expansion of the sum. As recurrence relation similar to (15) applies to $N_n'(x)$. The results are

$$J_n'(x) \sim \frac{A \sin(x - \frac{\pi}{4} - \frac{n\pi}{2})}{\sqrt{x}} + \varepsilon_3(x) ,$$

$$N_n'(x) \sim B \cos(x - \frac{\pi}{2} - \frac{n\pi}{2}) + \varepsilon_4(x) ,$$

where A and B are constants and $\varepsilon_3(x)$ and $\varepsilon_4(x)$ are both of the order of $x^{-3/2}$ and approach 0 as x becomes infinite.

Correction

The statement of the lemma on page 27 of special report No.170-1, while not actually incorrect, may be slightly misleading. It should be amended to read:

Let $f(\rho,x)$ and $\frac{\partial f(\rho,x)}{\partial x}$ be continuous for all $\rho > 0$ and tend to definite limits as ρ approaches 0, for all positive values of x. It then follows that $\frac{\partial f(0,x)}{\partial x} = \lim_{\rho \to 0} \frac{\partial f(\rho\ x)}{\partial x}$.

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